Math 259A Lecture 23 Notes

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1 Every II_1 Factor Has a Trace

This note is based on a set of slides Professor Popa used for the lecture.

1.1 Theorem and the hyperfinite II_1 factor

Last time, we defined the hyperfinite II_1 factor by constructing R_0 with the trace state τ. We can define the Hilbert space $L^2(R_0)$ as the completion of R_0 with respect to the Hilbert-norm $\|y\|_2 = \tau(y^*y)^{1/2}$, and denote $\widehat{R_0}$ as the copy of R_0 as a subspace of $L^2(R_0)$.

For each $x \in R_0$, define the operator $\lambda(x)$ on $L^2(R_0)$ by $\lambda(x)(\widehat{y}) = \widehat{xy}$ for all $y \in R_0$. Note that $x \mapsto \lambda(x)$ is a *-algebra morphism $R_0 \to \mathcal{B}(L^2)$ with $\|\lambda(x)\| = \|x\|$ for all x. Moreover, $\langle \lambda(x)(\hat{1}), \hat{1}\rangle_{L^2} = \tau(x)$.

We can similarly define $\rho(x)$ to be the right multiplication operator. Then λ and ρ commute. Last time, we showed that the von Neumann algebra $R = \overline{\lambda(R_0)}^{\mathrm{wo}}$ is a II_1 factor.

One other way to define R is as the completion of R_0 in the topology of convergence in hte norm $\|x\|_2 = \tau(x^*x)^{1/2}$ of sequences that are bounded in the operator norm. Notice that, in both definitions, τ extends to a trace state on R. If one denotes by $D_0 \subseteq R_0$ the natural "diagonal subalgebra," then $(D_0, \tau|_{D_0})$ coincides with the algebra of dyadic step functions on $[0, 1]$ with the Lebesgue integral. So its closure in R in the above topology, $(D, \tau|_D)$ is just $(L^\infty([0, 1]), \int \, d\mu)$.

Also, (R_0, τ) (and thus R) is completely determined by the sequence of partial isometries $v_1 = e^1_{1,2}$, $v_n = (\prod_{i=1}^{n-1} e^i_{2,2})e^n_{1,2}$ for $n \geq 2$ with $p_n = v_n v_n^*$; these satisfy $\tau(p_n) = 2^{-n}$ and $p_n \sim 1 - \sum_{i=1}^n p_i$.

Theorem 1.1. *Let M be a von Neumann factor. The following are equivalent:*

1. *M is a* ***finite*** *von Neumann algebra; i.e. if $p \in P(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (any isometry in M is necessarily a unitary element).*

2. *M has a* ***trace state*** *(i.e. a functional $\tau : M \to \mathbb{C}$ that is positive, $\tau(x^*x) \geq 0$, $\tau(1) = 1$, and $\tau(xy) = \tau(yx)$ for all $x, y \in M$).*

3. *M has a trace state τ that is **completely additive** (i.e. $\tau(\sum_i p_i) = \sum_i \tau p_i$) for for all mutually orthogonal projections $\mathcal{P}(M)$.*

4. *M has a trace state τ that is **normal** (i.e. $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ if $(x_i)_i \subseteq (M_+)_1$ is an increasing net).*

So a von Neumann factor is finite if and only if it is tracial. Moreover, such a factor has the unique trace state τ, which is automatically normal, faithful, and satisfies $\overline{\mathrm{co}}\{uxu^* : u \in U(M)\} \cap \mathbb{C}1 = \{\tau(x)1\}$ for all $x \in M$.

These are progressively stronger conditions, so we need only show that (4) $\implies$ (1). We need some lemmas.

1.2 Projections in a finite von Neumann factor

Lemma 1.1. *If a von Neumann factor M has a minimal projection, then $M = \mathcal{B}(\ell^2(I))$ for some I. Moreover, if $M = \mathcal{B}(\ell^2(I))$, then the following are equivalent:*

1. *M has a trace*

2. *$|I| < \infty$.*

3. *M is finite, i.e. if $u \in M$ with $u * u = 1$, then $uu^* = 1$.*

Proof. If we have a trace in finite dimensions, split $1 = p_1 + p_2$ into two projections onto infinite dimensional subspaces. Since trace is additive and $p_1 \sim p_2$, $\tau(p_1) = \tau(p_2) = 1$. Do the same with p_2 to get p_3 and p_4. But then $1 = p_1 + p_3 + p_4$, where $\tau(p_1) = \tau(p_3) = \tau(p_4)$ because these projections are equivalent. But this gives $\tau(p_1) = 1/3$, which is a contradiction. □

Lemma 1.2. *If M is finite, then*

1. *If $p, q \in P(M)$ are such that $p \sim q$, then $1 - p \sim 1 - q$.*

2. *pMp is finite for all $p \in P(M)$; i.e. if $q \in P(M)$ and $q \leq p$ with $q \sim p$, then $q = p$.*

Proof. Use the comparison theorem. □

Lemma 1.3. *If M is a von Neumann factor with no atoms (so $p \in P(M)$ has $\dim(pMp) = \infty$), then there exist $P_0, P_1 \in P(M)$ with $P_0 \sim P_1$ and $P_0 + P_1 = p$.*

So we can split p into two equivalent projections.

Proof. Consider the family $\mathcal{F} = \{(p_i^0, p_i^1)_i : p_i^0, p_j^1 \text{ mut. orth.}, \leq p, p_i^0 \sim p_i^1\}$ with the ordering from inclusion. Obtain a maximal element of $\mathcal{F}$. If $(p_i^0, p_i^1)_{i \in I}$ is a maximal element, then $P_0 = \sum_i p_i^0$ and $P_1 = \sum_o p_i^1$ will do; if not then the comparison theorem gives a contadiction. □

Lemma 1.4. *If M is a factor with no minimal projections, there exists a sequence of mutually orthogonal projections $(p_n)_n \subseteq P(M)$ such htat $p_n \sim 1 - \sum_{i=1}^n p_i$ for all n.*

Proof. Apply the previous lemma recursively. $\square$

Lemma 1.5. *If M is a finite factor and $(p_n)_n$ are as in the previous lemma, then*

1. *If $p \prec p_n$ for all n, then $p = 0$. Equivalently, if $p \neq 0$, there exists some n such that $p_n \prec p$. Moreover, if n is the first integer such that $p_n \prec p$ and $p'_n \leq p$ with $p'_n \sim p_n$, then $p - p'_n \prec p_n$.*

2. *If $(q_n)_n \subseteq P(M)$ is increasing, $q_n \leq q \in P(M)$, and $q - q_n \prec p_n$ for all n, then $q_n \nearrow q$ (with SO convergence).*

3. *$\sum_n p_n = 1$.*

Proof. If $p \sim p'_n \leq p_n$ for all n, then $P = \sum_n p'_n$, and $P_0 = \sum_k p'_{2k+1}$ satisfy $P_0 < P$ and $P_0 \sim P$. This contradicts the finiteness of M. $\square$

Lemma 1.6. *Let M be a finite factor without atoms. If $p \in P(M)$ is nonzero, then there is a unique infinite sequence $1 \leq n_1 < n_2 < \cdots$ such that p decomposes as $p = \sum_{k \geq 1} p'_{n_k}$ for some $(p_{n_k})_k \subseteq P(M)$ with $p'_{n_k} \sim p_{n_k}$ for all k.*

Proof. Apply part (1) of the previous lemma recursively. By part (2), the sum converges to p. $\square$

Definition 1.1. If M is a finite factor without atoms, the **dimension** is $\dim : P(M) \to [0,1]$ given by $\dim(p) = 0$ if $p = 0$ and $\dim(p) = \sum_{k=1}^{\infty} 2^{-n_k}$ if $p \neq 0$, where $n_1 < n_2 < \cdots$ are gven by the previous lemma.

Lemma 1.7. $\dim$ *satisfies the following conditions:*

1. $\dim(p_n) = 2^{-n}$.

2. *If $p, q \in P(M)$, then $p \leq q$ iff* $\dim(p) \leq \dim(q)$.

3. $\dim$ *is completely additive: if $q_i \in P(M)$ are mutually orthogonal, then* $\dim(\sum_i q_i) = \sum_i \dim(q_i)$.

1.3 The Radon-Nikodym trick

We claim that dim extends to the trace τ on $(M)_+$ in the following way. If $0 \leq x \leq 1$, then $x = \sum_{n=1}^{\infty} 2^{-n} e_n$. So if we put $\tau(x) = \sum 2^{-n} \dim(e_n)$, this is well-defined. Now if $x \in (M)_h$, we can take $\tau(x) = \tau(x_+) - \tau(x_-)$. And then we can extend this to M. But we have a problem; we cannot tell that this τ is linear.

Lemma 1.8 ("Radon-Nikodym trick"). *Let $\varphi, \psi : P(M) \to [0,1]$ be completely additive functions with $\varphi \neq 0$ and $\varepsilon > 0$. There exists a $p \in P(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$ and $\theta \geq 0$ such that $\theta\varphi(q) \leq \psi(q) \leq (1+\varepsilon)\theta\varphi(q)$ for all $q \in P(pMp)$.*

Intuitively, we want to think of φ, ψ like measures. In other words, we can take a small part of the space where φ and ψ are almost multiples of each other.

Proof. Denote $\mathcal{F} = \{p : \exists n \text{s.t.} p \sim p_n\}$. We may assume φ is faithful: take a maximal family of mutually orthogonal nonzero projections (e_i) with $\varphi(e_i) = 0$ for all i. Then let $f = 1 - \sum_i e_i \neq 0$ (because $\varphi(1) \neq 0$); it follows that φ is faithful on fMf, and by replacing with some $f_0 \leq f$ in $\mathcal{F}$, we may also assume $f \in \mathcal{F}$. Thus, proving the lemma for M is equivalent to proving it for fMF, which amounts to assuming φ is faithful.

If $\psi = 0$, then we take $\theta = 0$. If $\psi \neq 0$, then by replacing φ by $\varphi(1)^{-1}\varphi$ and ψ by $\psi(1)^{-1}\psi$, we may assume that $\varphi(1) = \psi(1) = 1$. We claim that this implies: There exists a $f \in \mathcal{F}$ such that for all $g_0 \in \mathcal{F}$ with $g_0 \leq g$, we have $\varphi(g_0) \leq \psi(g_0)$.

If not, then for all $g \in \mathcal{F}$, there is a $g_0 \in \mathcal{F}$ with $g_0 \leq g$ such that $\varphi(g_0) > \psi(g_0)$. Tkae a maximal family of mutually orthogonal projections $(g_i)_i \subseteq \mathcal{F}$ with $\varphi(g_i) > \psi(g_i)$ for all i. If $1 - \sum_i g_i \neq 0$, then take $g \in \mathcal{F}$ with $g \leq 1 - \sum_i g_i$ and apply this condition to get $g_0 \in \mathcal{F}$ with $g_0 \leq g$ and $\varphi(g_0) > \psi(g_0)$, contradicting maximality. Thus,

$$1 - \varphi\left(\sum_i g_i\right) = \sum_i \varphi(g_i) > \sum_i \psi(g_i) = \psi\left(\sum_i g_i\right) = \psi(1) = 1,$$

a contradiction. So this case is impossible.

Define $\theta = \sup\{\theta' : \theta'\varphi(g_0) \leq \psi(g_0) \; \forall g_0 \leq g, g_0 \in \mathcal{F}\}$. Then $1 \leq \theta < \infty$, and $\theta\varphi(g_0) \leq \psi(g_0)$ for all $g_0 \in \mathcal{F}$ with $g_0 \leq g$. Moreover, by definition of θ, there exists some $g_0 \in \mathcal{F}$ with $g_0 \leq g$ such that $\theta\varphi(g_0) > (1+\varepsilon)^{-1}\psi(g_0)$. We now repeat the argument for ψ and $\theta(1+\varepsilon)\varphi$ on g_0Mg_0 to prove the following:

We claim that there exists some $g' \in \mathcal{F}$ with $g' \leq g_0$ such that for all $g_0' \in \mathcal{F}$ with $g_0' \leq g_0$, we have $\psi(g_0') \leq \theta(1+\varepsilon)\varphi(g_0')$. If not, then for all $g' \in \mathcal{F}$ with $g' \leq g_0$, there is a $g_0' \leq g'$ in $\mathcal{F}$ such that $\psi(g_0') > \theta(1+\varepsilon)\varphi(g_0')$. But then take a maximal family of mutually orthogonal $g_i' \leq g_0$ such that $\psi(g_i') \geq \theta(1+\varepsilon)\varphi(g_i')$. Using one of the previous lemmas, we get $\sum_i g_i' = g_0$. Then $\psi(g_0) \geq \theta(1+\varepsilon)\varphi(g_0) > \psi(g_0)$. This is a contradiction. So the claim holds for some $g; \in \mathcal{F}$ with $g' \leq g_0$. Taking $p = g'$, we get that any $q \in \mathcal{F}$ under p satisfies both $\theta\varphi(q) \leq \psi(q)$ and $\psi(q) \leq \theta(1+\varepsilon)\varphi(q)$. By complete additivity of φ and ψ, using a previous lemma, we are done. □

Now apply the lemma to $\psi = \dim$ and φ to be a vector state on $M \subseteq \mathcal{B}(H)$ to get the following:

Lemma 1.9. *For all $\varepsilon > 0$, there exists some $p \in P(M)$ with $\dim(p) = 2^{-n}$ for some $n \geq 1$ and a vector state φ_0 on pMp such that for all $q \in P(pMp)$, $(1+\varepsilon^{-1}\varphi_0(q) \leq 2^n \dim(q) \leq (1+\varepsilon\varphi_0(q)$.*

We want to reproduce the linearity of the dimension function on pMp to the whole space.

Lemma 1.10. *With p, φ_0 as above, let $v_1 = p, v_2, \ldots, v_{2^n} \in M$ such that $v_i v_i^* = p$ and $\sum_i v_i v_i^* = 1$. Let $\varphi(x) := \sum_{i=1}^{2^n} \varphi_0(v_i x v_i^*)$ for $x \in M$. Then φ is a normal state on M satisfying $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$ for all $x \in M$.*

Proof. Note first that $\varphi_0(x^*x) \leq (1+\varepsilon)\varphi_0(xx^*)$ for all $x \in pMp$ (do it first for when x is a partial isometry, then for x with x^*x having finite spectrum). To deduce the inequality for φ itself, note that if $\sum_j v_j^* v_i = 1$, then for any $x \in M$,

$$\begin{aligned}
\varphi(x^*x) &= \sum_i \varphi_0\left(v_i x^* \left(\sum_j v_j^* v_j\right) x v_i^*\right) \\
&= \sum_{i,j} \varphi\varphi_0((v_i x^* v_j^*)(x_j x v_i)) \\
&\leq (1+\varepsilon^2) \sum_{i,j} \varphi_0((v_j x v_i)(v_i x^* v_j^*)) \\
&= \cdots \\
&= (1+\varepsilon^2)\varphi(xx^*).
\end{aligned}$$

□

Lemma 1.11. *If φ is a state on M that satisfies $\varphi(x^*x) \leq (1+\varepsilon)\varphi(xx^*)$ for all $x \in M$, then $(1+\varepsilon)^{-1}\varphi(p) \leq \dim(p)(1+\varepsilon)\varphi(p)$ for all $p \in P(M)$.*

Proof. By complete additivity, it is sufficient to prove it for $p \in \mathcal{F}$, when we have $v_1, \ldots, v_{2^n}$ as in the previous lemma. Then $\varphi(p) = \varphi(v_j^* v_j) \leq (1+\varepsilon)\varphi(v_j v_j^*)$ for all j, so

$$2^n \varphi(p) \leq (1+\varepsilon)^2 \sum_j \varphi(v_j v_j^*) = (1+\varepsilon)^2 2^n \dim(p).$$

Similarly, $2^n \dim(p) = 1 \leq (1+\varepsilon)^2 2^n \varphi(p)$. □

1.4 Proof of the theorem

Now we can prove the theorem.

Proof. Define τ as mentioned before. By the previous lemma, for every $\varepsilon > 0$, there is a normal state φ on M such that $|\tau(p) - \varphi(p)| \leq \varepsilon$ for all $p \in P(M)$. By definition of $|tau$ and the linearity of φ, this implies that $|\tau(x) - \varphi(x)| \leq \varepsilon$ for all $x \in (M_+)_1$. So $|\tau(x) - \varphi(x)| \leq 4\varepsilon$ for all $x \in (M)_1$. This implies that $\tau(x+y) - \tau(x) - \tau(y)| \leq 8\varepsilon$ for all $x, y \in (M)_1$. Since $\varepsilon > 0$ was arbitrary, we get that τ is a linear state on M.

By definition of τ, we also have $\tau(uxu^*) = \tau(x)$ for all $x \in M$ and $u \in U(M)$. So τ is a trace state. From the above argument it also follows that norm limit of normal states φ, so τ is normal as well. □

This theorem also has a generalization.

Theorem 1.2. *Let M be a von Neumann algebra that is countably decomposable (i.e. any family of mutually orthogonal projections is countable). The following are equivalent:*

1. *M is a* ***finite*** *von Neumann algebra; i.e. if $p \in P(M)$ satisfies $p \sim 1 = 1_M$, then $p = 1$ (so any isometry in M is necessarily a unitary element).*

2. *M has a faithful, normal (equivalently completely additive) trace state τ.*

Moreover, if M is finite, then there exists a unique normal faithful ***central trace****, i.e. a linear positive map $\mathrm{ctr} : M \to Z(M)$ that satisfies $\mathrm{ctr}(1) = 1$, $\mathrm{ctr}(z_1 x z_2) = z_1 \mathrm{ctr}(x) z_2$, and $\mathrm{ctr}(xy) = \mathrm{ctr}(yx)$ for al $lx, y \in M$ and $z_i \in Z$.*

Any trace τ on M is of the form $\tau = \varphi_0 \circ \mathrm{ctr}$ for some state φ on Z. Also, $\overline{\mathrm{co}}\{uxu^* : u \in U(M)\} \cap Z = \{\mathrm{ctr}(x)\}$ for all $x \in M$.

The central trace should be thought of like a conditional expectation onto $Z(M)$.